FCS Seminar 7: Dynamical Systems Analysis

The idea of this problem sheet is to introduce some techniques used to analyse dynamical systems. The aim of the techniques is to try to determine the long-term behaviour of the system, what states it is likely to be in and whether it can move between them. By the end you will be able to:

1. Find fixed points of a dynamical system
2. Investigate the stability of fixed points
3. Use cobweb plots to analyse dynamical systems
4. Use the phase-plane to analyse dynamical systems

Questions

1. *Exponential growth/decay* ***[2 marks]:*** Suppose we have the following equation:

$$\frac{dx}{dt} = kx \tag{1}$$

where x is population size and k governs the rate of growth, which we are to integrate using Euler integration ie using:

$$x(t+h) = x(t) + h\frac{dx}{dt} \tag{2}$$

To find and analyse fixed points of this system (and to make its dependence on h explicit) we must re-formulate equations 1 and 2 so that $x(t+h) = f(x(t))$ for some function f.

   (a) Use equations 1 and 2 to write down what f is in this case.

   (b) Find the fixed point of the system (Tips: fixed points of a dynamical system where $x(t+h) = f(x)$ can be found by setting $x = f(x)$ and solving eg if the system was $x(t+h) = f(x(t)) = (x(t))^2$, we would set $x = x^2$ and solve to get $x = 1, x = -1$ or $x = 0$).

   (c) Denoting the fixed point as a_0, determine the stability of the system for $k > 0$ by evaluating $f'(a_0)$ ie calculate $\frac{df}{dx}$ and evaluate it at a_0. Remember that $h > 0$. (Tips: the system is stable if $|f'(a_0)| < 1$, unstable if $|f'(a_0)| > 1$ and inconclusive if $|f'(a_0)| = 1$ eg the system $x(t+h) = f(x(t)) = (x(t))^2$ has fixed points $a_0 = 0, \pm 1$. $f'(x) = \frac{df}{dx} = 2x$ so $|f'(a_0)| = |2a_0| = 0$ for $a_0 = 0$, which is therefore stable, and 2 for $a_0 = \pm 1$ which are both unstable)

   (d) Investigate the stability for $k < 0$. If h is fixed, what range of values of k lead to a stable system and what to an unstable system (note that the range will be in terms of h). What is the relationship needed between h and k to make $|f'(a_0)| = 1$ where a_0 is the fixed point of the system (and is the same as found in question 1c). Using your program from last week's seminar, see what happens to x if you use $k = -2$, $x(0) = 1$ and run it for a simulation length of $100s$ using different values for h. First use $h = 0.75s$, then use the value of h needed to make $|f'(a_0)| = 1$, which we shall call h_0. Finally, use $h = h_0 + 0.01$ then $h = h_0 - 0.01$. Note how much the long-term behaviour changes given such a small change in h. We therefore refer to $h = h_0$ as a *bifurcation point* of the system.

2. *GasNet, single neuron* ***[4 marks]:*** A GasNet neuron with positive self-recurrency and external input I updated iteratively is a discrete dynamical system governed by the following equation:

$$x(n+1) = f(x(n)) = \tanh\left(C\left(x(n) + I\right)\right)$$

where $x(n)$ is the neuron output at time-step n. C is a parameter which governs the slope of the transfer function tanh. Its default value is genetically-specified, but C is altered temporarily in the presence of a gaseous neuromodulator which may be emitted by other neurons.

(a) Suppose $I = 0$. By plotting $y = x$ and $y = f(x)$ between -2 and 2 (eg using x=-2:0.01:2 and the matlab function 'tanh') on the same graph, find (approximately) the fixed points of the system for the following values of C: $0.5, 1, 2, -0.9, -1$ and -1.1. Examine the stability of the fixed points by plotting $f'(x) = C(sech(Cx))^2$ between -2 and 2 (matlab has the function 'sech'). By examining the graphs and equations, deduce what range of C values lead to 3 fixed points rather than 1 and the stability of the resultant points.

Check your results using cobweb plots and plotting x over time, using the code fragments contained in GasNetEGs.m. Start from different values of x and examine the behaviour of the systems. Note the range of different behaviours for $C < 0$.

(b) Now set $C = 1.5$ and $I = 0.1$ and examine the fixed points and their stability using the techniques of question 2a. How do the number and behaviour of the fixed points change as I is increased from 0.1 to 0.15? Start both systems at $x(0) = -0.9$ and plot x against time to see the differences between the systems.

(c) In an evolved GasNet network, neuron A has default $C = 0.5$ and no external input ie $I = 0$ and governs a robot's wheel-motor. It is influenced by gas from neuron B, which it has an inhibitory electrical connection to. Neuron B emits gas continuously but at a low concentration, meaning that Neuron A's C-value remains unchanged until the gas has built up for 5 time-steps. At this point, the gas concentration crosses a threshold value and neuron A's C value at subsequent time-steps is changed to -2. This situation continues until neuron A's output $x \geq 0.95$. This inhibits neuron B strongly, stopping gas emission, so that A's C-value returns to 0.5, and the gas concentration returns to zero.

Change your program to include a gas concentration parameter and 2 'if' clauses. The 1st checks whether x is ≥ 0.95, setting the gas concentration to 0 if it is, and incrementing the gas concentration by 1 if not. The second checks whether gas concentration is ≥ 5. If it is, change neuron A's C-value as specified, if not, revert to the old value. Iterate through this system starting from an initial value of $x = 1$ for neuron A and zero gas concentration and observe the motor output. How would you describe the resultant behaviour?

3. *GasNet analysis, 2 neurons [4* ***marks****]:* Suppose we have a network of 2 GasNet neurons, A and B. Neuron A has default $C = 0.5$, positive self-recurrency and external input from a light sensor $I(n)$. Neuron B has $C = 1.2$, positive self-recurrency and receives inhibitory input from A. Denoting the outputs of A and B at the n'th time-step as $x(n)$ and $y(n)$ respectively, the governing equations are:

$$x(n+1) = f(x(n), y(n)) = \tanh\left(0.5\left(x(n) + I(n)\right)\right) \quad (3)$$

$$y(n+1) = g(x(n), y(n)) = \tanh\left(1.2\left(y(n) - x(n)\right)\right) \quad (4)$$

(a) Assuming $I(n) = 0$ determine the fixed point of neuron A, x_0, by plotting $x = x$ and $x = f(x)$. Use this value to plot $y = g(x_0, y)$ and thus determine the fixed points of the system (remember that at fixed points of the system, both neurons must be at a fixed point).

(b) Write a program that plots x and y over time and x against y (ie a phase-plane plot of x and y) using the code fragments in GasNetEGs.m. Use the program to investigate the behaviour of the system starting from different initial values of x and y (eg $x = -0.5$ and $y = 0$, $x = -0.5$ and $y = 0.5$ etc) and thus determine the stability of the fixed points.

(c) Suppose a light starts flashing so that every 10 time-steps the light sensor inputs 0.5 to neuron A. Alter the program so that $I(n) = 0.5$ every 10 time-steps (note 'mod(n,10)' returns a zero every time n is a multiple of 10). What happens if you start at $x = -0.5$ and $y = 0.5$ and run the system for 100 iterations?

(d) Suppose now that the light is switched off, but that every 5th time-step neuron B is affected by gas so that for that time-step its C-value switches from $C = 1.2$ to $C = -1.2$. Again investigate the behaviour of the system starting from $x = -0.5$ and $y = 0.5$.

(e) Finally, investigate the behaviour of the system starting from $x = -0.5$ and $y = 0.5$ if the light switched back on ie so that every 5th time-step the C-value of B switches from $C = 1.2$ to $C = -1.2$ and every 10th time-step $I(n) = 0.5$.